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A comprehensive note on "Lot-sizing decisions for deteriorating items with two warehouses under an order-size-dependent trade credit"

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Abstract

To reduce inventory and increase sales, the supplier frequently offers the retailer a permissible delay in payments if the retailer orders more than or equal to a predetermined quantity. In 2012, Liao et al. proposed an economic order quantity model for a retailer with two warehouses when the supplier offers a permissible delay linked to order quantity. In this paper, we attempt to overcome some shortcomings of their mathematical model. Then, we apply some existing theoretical results in fractional convex programs to prove that the annual total variable cost is pseudoconvex. Hence, the optimal solution exists uniquely, which simplifies the search for the global minimum solution to a local minimum solution. Finally, we run a couple of numerical examples to illustrate the problem and compare the optimal solutions between theirs and ours.

Keywords: inventory theory; economic order quantity; deteriorating items; two warehouses; trade credit

1. Introduction

In order to increase sales and reduce inventory, the seller (e.g., the supplier) frequently offers a permissible delay in payments (i.e., a trade credit) to the buyer (e.g., the retailer) if the buyer's order quantity is greater than or equal to a predetermined quantity by the seller. Usually, there is no interest charge if the buyer pays in full within the credit period. However, if the buyer cannot pay in full by the credit period, then the seller starts charging the buyer interest on unpaid balance after the credit period. Haley and Higgins (1973) first studied the impact of the trade credit

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financing on the inventory lot-sizing policy. Kingsman (1983) discussed the effect of various trade credit rules on ordering and stocking in purchasing. Goyal (1985) established an economic order quantity (EOQ) model under conditions of permissible delay in payments. Thereafter, most recent researchers in the field of inventory lot-sizing policies with trade credit financing have extended his basic model. Ouyang et al. (2006) presented an inventory model for noninstantaneous deteriorating items with permissible delay in payments. Teng et al. (2006) studied manufacturer's optimal pricing and lot-sizing policies under trade credit financing. Yang and Wee (2006) established a collaborative inventory model for deteriorating items with permissible delay in payments, finite replenishment rate, and price-sensitive demand. Chang et al. (2009) developed an integrated inventory model when trade credit linked to order quantity. Liao and Huang (2010) proposed an EOQ model for deteriorating items with trade credit financing and capacity constraints. Dye and Ouyang (2011) studied a retailer's optimal pricing and lot-sizing problem for deteriorating items with a fluctuating demand under trade credit financing. Ho (2011) developed a generalized, integrated, supplier–retailer inventory model using a trade credit policy. Liang and Zhou (2011) provided a two-warehouse inventory model for deteriorating items under conditionally permissible delay in payments. Teng et al. (2011) considered optimal ordering policy for stock-dependent demand under progressive payment scheme. Duan et al. (2012) established two-level supply chain coordination with permissible delay in payments for fixed lifetime products. Jain and Aggarwal (2012) studied an inventory model for exponentially deteriorating and imperfect quality items when a trade credit is offered by the supplier. Sarkar (2012) considered an EOQ model with delay in payments and time-varying deterioration rate. Teng et al. (2012) extended the traditional constant-demand EOQ model with trade credit financing to nondecreasing demand. Many related recent articles can be found in Chen et al. (2013a, 2013b), Chern et al. (2013), among others.

Recently, Liao et al. (2012) generalized Goyal's EOQ model to allow for deteriorating items with two warehouses (i.e., an owned warehouse (OW) with the maximum storage capacity of W, and a rented warehouse (RW) with unlimited storage capacity) under an order size dependent trade credit. In this paper, we attempt to overcome some shortcomings in their model such as (1) the cost of deteriorating items was included twice in their objective function, (2) we substitute the unit purchase cost by unit selling price in calculations of interest earned, and (3) they erroneously considered the time taken by inventory in RW to reduce to zero, to be same as the time taken by total inventory to reduce to the storage capacity of the OW. Then we apply some existing theoretical results in fractional convex programs to prove the annual total variable cost is pseudoconvex. Therefore, the optimal solution to the problem not only exists but is also unique. Finally, we run a couple of numerical examples to illustrate the problem.

2. Notation and assumptions

For convenience, the following notation and assumptions are used throughout the paper.

A: the rental cost for renting an additional warehouse in dollars
C: the unit purchase cost in dollars
D: the annual demand rate in units per year

h: the unit holding cost excluding interest charges per year in dollars for items in OW
I: the earned interest rate per dollar per year (as a percentage)
$I(t)$: the inventory level in units at time t in both RW and OW, $0 \le t \le T$
k: the unit holding cost excluding interest charges per year in dollars for items in RW
M: the credit period in years provided by the supplier to the retailer
P: the unit selling price in dollars, with $C < P$
Q: the order quantity in units
R: the capital opportunity cost per dollar per year (as a percentage)
S: the ordering cost per order in dollars
t: the time in years
t_R: the time in years when inventory level in RW reduces to zero
t_W: the time in years when inventory level reduces to W
T: the replenishment cycle time in years
$\overline{W}$: the quantity in units at which the delay in payments is permitted
W: the maximum storage capacity in units in OW
λ: a constant deterioration rate with $0 \le \lambda < 1$

In addition, the following assumptions are made to establish the mathematical inventory model.

(1) Replenishment rate is infinite and lead time is zero.
(2) If $Q < \overline{W}$, then the retailer must pay the supplier in full as soon as items are received. Otherwise, there is no interest charge if the outstanding amount is paid within the permissible delay period M.
(3) During the permissible delay, the retailer deposits the sales revenue to an interest bearing account with interest rate I. If $M \ge T$, then the retailer receives all revenue and pays off the entire purchase cost at the end of the permissible delay M. Otherwise (if $M < T$), the retailer pays the supplier all units sold by M, keeps the profit for the use of the other activities, and starts paying for the interest charges on the items sold after M.
(4) The OW has a fixed capacity of W units and the RW has unlimited capacity. The items in RW are consumed first, and then the items in OW.
(5) In today's time-based competition, we assume that shortages are not allowed.

Given the above assumptions, it is possible to formulate a mathematical inventory model for deteriorating items with two warehouses and trade credit financing.

3. Mathematical model

The units purchased include both units sold (i.e., good items) and units unsold (i.e., deteriorating) items. Consequently, the purchase cost includes the cost of deteriorating items. Hence, we define the annual total variable cost function as follows:

$$\begin{aligned} TVC(T) = {} & \text{ordering cost} + \text{purchase cost} + \text{stock-holding cost in RW} \\ & + \text{stock-holding cost in OW} + \text{capital opportunity cost} + \text{rental cost in RW.} \end{aligned} \tag{1}$$

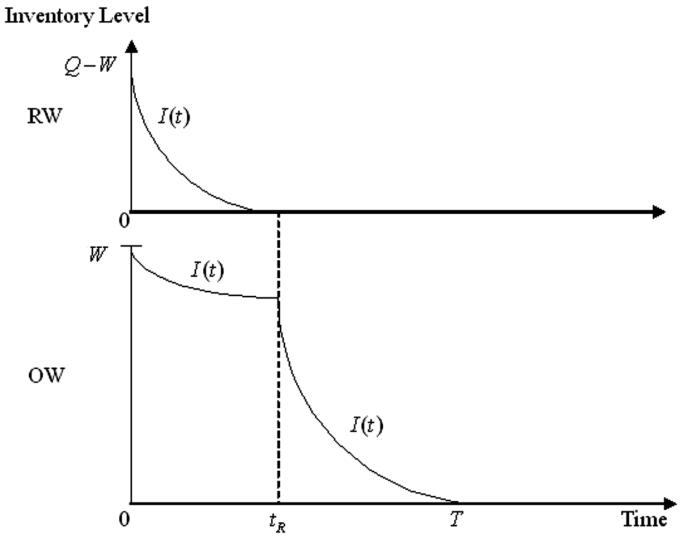

Fig. 1. Graphical representation of a two-warehouse inventory ($Q > W$).

By contrast, Liao et al. (2012) inappropriately calculated the cost of deteriorating items twice, and hence defined the annual total variable cost function as follows:

$$\begin{aligned} TVC(T) = {} & \text{ordering cost} + \text{purchase cost} + \text{deterioration cost} \\ & + \text{stock-holding cost in RW} + \text{stock-holding cost in OW} \\ & + \text{capital opportunity cost} + \text{rental cost in RW}. \end{aligned} \tag{2}$$

The order quantity Q units are ordered and received at $t = 0$. If $Q > W$, then an RW is used to store $Q - W$ units, and the OW has W units at $t = 0$. The inventory level in RW then gradually depletes to zero at $t = t_R$ due to the combination effects of demand and deterioration. The graphical representation of the inventory level in both warehouses is shown in Fig. 1.

During the replenishment cycle [0, T], the inventory level at each warehouse is depleted by deterioration and demand. We discuss the inventory level in RW first, and then that in OW. The inventory level in RW can be represented using the following differential equation:

$$\frac{dI(t)}{dt} = -\lambda I(t) - D, \quad 0 \le t \le t_R, \tag{3}$$

with the boundary conditions $I(t_R) = 0$ and $I(0) = Q - W$. The solution of Equation (3) is

$$I(t) = \frac{D}{\lambda}\left[e^{\lambda(t_R - t)} - 1\right], \quad 0 \le t \le t_R. \tag{4}$$

Next, the inventory level in OW can be represented using the following differential equations:

$$\frac{dI(t)}{dt} = -\lambda I(t), \quad 0 \le t \le t_R, \tag{5}$$

and

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\lambda I(t) - D, \quad t_R \le t \le T, \tag{6}$$

with the boundary conditions $I(0) = W$ and $I(T) = 0$. The solution of Equation (5) is

$$I(t) = W\mathrm{e}^{-\lambda t}, \quad 0 \le t \le t_R. \tag{7}$$

The solution of Equation (6) is

$$I(t) = \frac{D}{\lambda}\left[\mathrm{e}^{\lambda(T-t)} - 1\right], \quad t_R \le t \le T. \tag{8}$$

Using Equations (7) and (8), the solution of t_R is obtained as

$$t_R = \frac{1}{\lambda}\ln\left[\mathrm{e}^{\lambda T} - \frac{\lambda}{D}W\right]. \tag{9}$$

Using Equations (4) and (9), and the fact that the initial inventory level in RW is $I(0) = Q - W$, we obtain the order size

$$Q = \frac{D}{\lambda}(\mathrm{e}^{\lambda T} - 1). \tag{10}$$

Therefore, the annual stock-holding cost in RW is as follows:

$$\begin{aligned}\frac{k}{T}\int_0^{t_R} I(t)\mathrm{d}t &= \frac{k}{T}\int_0^{t_R} \frac{D}{\lambda}[\mathrm{e}^{\lambda(t_R-t)} - 1]\mathrm{d}t = \frac{kD}{T\lambda^2}\left[\mathrm{e}^{\lambda t_R} - \lambda t_R - 1\right] \\ &= \frac{kD}{T\lambda^2}\left[\mathrm{e}^{\lambda T} - \frac{\lambda}{D}W - \ln(\mathrm{e}^{\lambda T} - \lambda W/D) - 1\right].\end{aligned} \tag{11}$$

Note that Liao et al. (2012) inappropriately used the time t_W (i.e., when the inventory level in both warehouses reduces to W) to indicate the inventory level in RW is down to zero. Consequently, they miscalculated the annual stock-holding cost in RW from 0 to t_W, instead of from 0 to t_R, as follows:

$$\frac{k}{T}\int_0^{t_W} (I(t) - W)\mathrm{d}t = \frac{k}{\lambda^2 T}[D(\mathrm{e}^{\lambda T} - \mathrm{e}^{\lambda T_a}) - (D\lambda + \lambda^2 W)(T - T_a)], \tag{12}$$

where $T_a = \frac{1}{\lambda}\ln(\frac{\lambda}{D}W + 1)$ and $t_W = T - T_a$.

Similarly, the annual stock-holding cost in OW is as follows:

$$\begin{aligned}\frac{h}{T}\left[\int_0^{t_R} I(t)\mathrm{d}t + \int_{t_R}^{T} I(t)\mathrm{d}t\right] &= \frac{h}{T}\left[\int_0^{t_R} W\mathrm{e}^{-\lambda t}\mathrm{d}t + \int_{t_R}^{T} \frac{D}{\lambda}(\mathrm{e}^{\lambda(T-t)} - 1)\mathrm{d}t\right] \\ &= \frac{h}{T}\left[\frac{W}{\lambda} - \frac{D}{\lambda}T + \frac{D}{\lambda^2}\ln(\mathrm{e}^{\lambda T} - \lambda W/D)\right].\end{aligned} \tag{13}$$

Again, the mathematical expression of the annual stock-holding cost in OW in Liao et al. (2012) was also inappropriate as follows:

$$\frac{h}{T}\left(Wt_w + \int_{t_W}^{T} I(t)\mathrm{d}t\right) = \frac{h}{\lambda^2 T}\left[D\left(\mathrm{e}^{\lambda T_a} - \lambda T_a - 1\right) + \lambda^2 W\left(T - T_a\right)\right]. \tag{14}$$

After discussing the differences between Liao et al. (2012) and ours on the objective function and stock-holding costs, we then discuss the discrepancies between theirs and ours on the annual capital costs. If $Q \geq \overline{W}$, then the delay in payments is permitted. There are two possible alternatives for the retailer to select its replenishment cycle time T as follows: (1) $M < T$, or (2) $M \geq T$. Let us discuss the first alternative $M < T$ first, and then $M \geq T$.

If $M < T$, $Q > W$, and $M \leq t_R \leq T$, then the annual capital opportunity cost is as follows:

$$\frac{1}{T}\left\{CR\int_M^T I(t)\mathrm{d}t - PI\int_0^M Dt\mathrm{d}t\right\} = \frac{CRD}{\lambda^2 T}\left[\mathrm{e}^{\lambda(T-M)} - \lambda(T-M) - 1\right] - \frac{PIDM^2}{2T}. \tag{15}$$

Note that Liao et al. (2012) used the unit purchase cost C to calculate the revenue received, and then derived the interest earned. As a result, their annual capital opportunity cost for $M < T$ was as follows:

$$\frac{CRD}{\lambda^2 T}\left[\mathrm{e}^{\lambda(T-M)} - \lambda(T-M) - 1\right] - \frac{CIDM^2}{2T}. \tag{16}$$

If $M \geq T$, then there is no interest charged, and the annual capital opportunity cost is

$$\frac{-1}{T}PI\left[\int_0^T Dt\mathrm{d}t + DT(M-T)\right] = \frac{PIDT}{2} - PIDM. \tag{17}$$

Likewise, the mathematical expression of the annual capital opportunity cost in Liao et al. (2012) was as follows:

$$\frac{CIDT}{2} - CIDM. \tag{18}$$

Finally, we use a simpler and easier way to discuss all possible nine subcases than those 21 subcases in Liao et al. (2012). Based on the values of parameters W and $\overline{W}$, there are two possible cases: either $W \geq \overline{W}$ or $W < \overline{W}$. We discuss them accordingly.

Case 1: $W \geq \overline{W}$

In this case, there are three possible alternatives for the retailer to select its order quantity Q: $Q > W$, $W \geq Q \geq \overline{W}$, and $Q < \overline{W}$. Let us discuss them separately.

Subcase 1–1: $Q > W$

In this subcase, the RW is needed and the delay in payments is also permitted. The annual total variable costs are given as follows:

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if } M \leq T \\ TVC_2(T) & \text{if } M \geq T \end{cases}, \tag{19}$$

where

$$TVC_1(T) = \frac{S+A}{T} + \frac{D(C\lambda + k)}{\lambda^2 T}(e^{\lambda T} - 1) - \frac{D(k-h)}{\lambda^2 T}\left[\frac{\lambda}{D}W + \ln(e^{\lambda T} - \lambda W/D)\right] - \frac{hD}{\lambda} + \frac{CRD}{\lambda^2 T}\left[e^{\lambda(T-M)} - \lambda(T-M) - 1\right] - PI\frac{DM^2}{2T}, \tag{20}$$

and

$$TVC_2(T) = \frac{S+A}{T} + \frac{D(C\lambda + k)}{\lambda^2 T}(e^{\lambda T} - 1) - \frac{D(k-h)}{\lambda^2 T}\left[\frac{\lambda}{D}W + \ln(e^{\lambda T} - \lambda W/D)\right] - \frac{hD}{\lambda} - PID\left(M - \frac{T}{2}\right). \tag{21}$$

In contrast, the corresponding $TVC_1(T)$ and $TVC_2(T)$ in Liao et al. (2012) were

$$\frac{S+A}{T} + \frac{CD}{\lambda T}(e^{\lambda T} - 1) + \frac{D(\lambda C + k)}{\lambda^2 T}(e^{\lambda T} - \lambda T - 1) - \frac{(k-h)}{\lambda^2 T}\left[D\left(e^{\lambda T_a} - \lambda T_a - 1\right) + \lambda^2 W(T - T_a)\right] + \frac{CRD}{\lambda^2 T}\left[e^{\lambda(T-M)} - \lambda(T-M) - 1\right] - CI\frac{DM^2}{2T}$$

and

$$\frac{S+A}{T} + \frac{CD}{\lambda T}(e^{\lambda T} - 1) + \frac{D(\lambda C + k)}{\lambda^2 T}(e^{\lambda T} - \lambda T - 1) - \frac{(k-h)}{\lambda^2 T}[D(e^{\lambda T_a} - \lambda T_a - 1) + \lambda^2 W(T - T_a)] - CID\left(M - \frac{T}{2}\right),$$

respectively. By comparing theirs and ours, one can easily see that the differences are significantly large because the following three major reasons: (R1) They double counted the cost of deteriorating items in their objective function; (R2) They calculated the interest earned based on the unit purchase cost C, not the unit selling price P; and (R3) They erroneously considered the time taken by inventory in RW to reduce to zero, to be same as the time taken by total inventory in both OW and RW to reduce to the maximum capacity in OW, W. In fact, the inventory level in OW deteriorates from W to below W, whereas the inventory level in RW reduces from $Q - W$ to zero. Hence, when the inventory level in RW reduces to zero, the total inventory level is less than W.

Subcase 1–2: $W \geq Q \geq \overline{W}$

In this subcase, the RW is not needed, whereas the delay in payments is permitted. Therefore, the annual total variable cost $TVC(T)$ is obtained as follows:

$$TVC(T) = \begin{cases} TVC_3(T) & \text{if } M \leq T \\ TVC_4(T) & \text{if } M \geq T \end{cases}, \tag{22}$$

where

$$TVC_3(T) = \frac{S}{T} + \frac{C}{T}\frac{D}{\lambda}(e^{\lambda T} - 1) + \frac{h}{T}\frac{D}{\lambda^2}[e^{\lambda T} - \lambda T - 1] + \frac{CRD}{\lambda^2 T}\left[e^{\lambda(T-M)} - \lambda(T-M) - 1\right] - PI\frac{DM^2}{2T}, \tag{23}$$

and

$$TVC_4(T) = \frac{S}{T} + \frac{C}{T}\frac{D}{\lambda}(e^{\lambda T} - 1) + \frac{hD}{\lambda^2 T}[e^{\lambda T} - \lambda T - 1] - PID\left(M - \frac{T}{2}\right). \tag{24}$$

By contrast, the corresponding $TVC_3(T)$ and $TVC_4(T)$ in Liao et al. (2012) were

$$\frac{S}{T} + \frac{C}{T}\frac{D}{\lambda}(e^{\lambda T} - 1) + \frac{D(\lambda C + h)}{\lambda^2 T}(e^{\lambda T} - \lambda T - 1) + \frac{CRD}{\lambda^2 T}[e^{\lambda(T-M)} - \lambda(T - M) - 1] - \frac{CIDM^2}{2T},$$

and

$$\frac{S}{T} + \frac{C}{T}\frac{D}{\lambda}(e^{\lambda T} - 1) + \frac{D(\lambda C + h)}{\lambda^2 T}(e^{\lambda T} - \lambda T - 1) - DCI\left(M - \frac{T}{2}\right)$$

respectively. We then discuss the last subcase in which $Q < \overline{W}$.

Subcase 1–3: $Q < \overline{W}$

In this subcase, the RW is not needed and the delay in payments is not permitted. Therefore, the annual total variable cost $TVC(T)$ is obtained as follows:

$$TVC_5(T) = \frac{S}{T} + \frac{C}{T}\frac{D}{\lambda}(e^{\lambda T} - 1) + \frac{D(h + CR)}{\lambda^2 T}[e^{\lambda T} - \lambda T - 1]. \tag{25}$$

Likewise, the corresponding $TVC_5(T)$ in Liao et al. (2012) was

$$\frac{S}{T} + \frac{C}{T}\frac{D}{\lambda}(e^{\lambda T} - 1) + \frac{D(\lambda C + h + CR)}{\lambda^2 T}[e^{\lambda T} - \lambda T - 1].$$

Now, Case 1 is finished. Next, we discuss Case 2 of $W < \overline{W}$.

Case 2: $W < \overline{W}$

Similar to Case 1, there are three possible alternatives for the retailer to select its order quantity Q: $Q \geq \overline{W}$, $W < Q < \overline{W}$, and $Q \leq W$. Let us discuss them accordingly.

Subcase 2–1: $Q \geq \overline{W}$

In this subcase, the RW is needed and the delay in payments is also permitted. Consequently, the annual total variable cost $TVC(T)$ is obtained as follows:

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if } M \leq T \\ TVC_2(T) & \text{if } M \geq T \end{cases}. \tag{26}$$

Subcase 2–2: $W < Q < \overline{W}$

In this subcase, the RW is needed, whereas the delay in payments is not permitted. Therefore, the annual total variable cost $TVC(T)$ is obtained as follows:

$$TVC_6(T) = \frac{S + A}{T} + \frac{D(C\lambda + k)}{\lambda^2 T}(e^{\lambda T} - 1) - \frac{D(k - h)}{\lambda^2 T}\left[\frac{\lambda}{D}W + \ln(e^{\lambda T} - \lambda W/D)\right] - \frac{hD}{\lambda} + \frac{CRD}{\lambda^2 T}[e^{\lambda T} - \lambda T - 1]. \tag{27}$$

Similarly, the corresponding $TVC_6(T)$ in Liao et al. (2012) was

$$\frac{S+A}{T}+\frac{CD}{\lambda T}(\mathrm{e}^{\lambda T}-1)+\frac{D(\lambda C+h+CR)}{\lambda^2 T}[\mathrm{e}^{\lambda T}-\lambda T-1]$$
$$-\frac{(k-h)}{\lambda^2 T}\left[D(\mathrm{e}^{\lambda T_a}-\lambda T_a-1)+\lambda^2 W(T-T_a)\right].$$

Subcase 2–3: $Q \le W$

In this subcase, the RW is not needed and the delay in payments is not permitted. Hence, the annual total variable cost $TVC(T) = TVC_5(T)$.

Now, we determine the optimal replenishment cycle T^* for cases of $W \ge \overline{W}$, and $W < \overline{W}$. In the next section, we show that $TVC_i(T)$ for $i = 1, 2, \ldots, 6$ is a strictly pseudoconvex function in T.

4. Optimal replenishment cycle time

To solve the problem, we apply the existing theoretical results. According to Theorems 3.2.9, and 3.2.10 in Cambini and Martein (2009), the real-value function

$$q(x) = \frac{f(x)}{g(x)} \tag{28}$$

is (strictly) pseudoconvex, if $f(x)$ is nonnegative, differentiable and (strictly) convex, and $g(x)$ is positive, differentiable and concave. Furthermore, if $\nabla q(x_0) = 0$, then x_0 is a local minimum for q. Now, let us apply the above theoretical results to show that the optimal solution T_i^* that minimizes $TVC_i(T)$ for $i = 1, 2, \ldots, 6$ not only exists but is also unique.

Theorem 1. *$TVC_i(T)$, for $i = 1, 2, \ldots, 6$, is a strictly pseudoconvex function in T, and hence exists a unique minimum solution T_i^*, for $i = 1, 2, \ldots, 6$.*

Proof. From Equation (20), let

$$f_1(T) = S + A + \frac{D(C\lambda+k)}{\lambda^2}(\mathrm{e}^{\lambda T}-1) - \frac{D(k-h)}{\lambda^2}\left[\frac{\lambda}{D}W + \ln(\mathrm{e}^{\lambda T} - \lambda W/D)\right]$$
$$-\frac{hD}{\lambda}T + \frac{CRD}{\lambda^2}[\mathrm{e}^{\lambda(T-M)} - \lambda(T-M) - 1] - PI\frac{DM^2}{2} \tag{29}$$

and

$$g_1(T) = T. \tag{30}$$

Taking the first-order and second-order derivatives of $f_1(T)$, we have

$$f_1'(T) = \frac{D(C\lambda+k)}{\lambda}\mathrm{e}^{\lambda T} - \frac{D(k-h)}{\lambda}\frac{\mathrm{e}^{\lambda T}}{\mathrm{e}^{\lambda T}-\lambda W/D} - \frac{hD}{\lambda} + \frac{CRD}{\lambda}[\mathrm{e}^{\lambda(T-M)}-1], \tag{31}$$

and

$$f_1''(T) = D(C\lambda+k)\mathrm{e}^{\lambda T} + \frac{D(k-h)(\lambda W/D)\mathrm{e}^{\lambda T}}{(e^{\lambda T}-\lambda W/D)^2} + CRD\mathrm{e}^{\lambda(T-M)} > 0. \tag{32}$$

Therefore, $q_1(T) = \frac{f_1(T)}{g_1(T)} = TVC_1(T)$ is a strictly pseudoconvex function in T. Similarly, from Equation (21), let

$$f_2(T) = S + A + \frac{D(C\lambda + k)}{\lambda^2}(e^{\lambda T} - 1) - \frac{D(k-h)}{\lambda^2}\left[\frac{\lambda}{D}W + \ln(e^{\lambda T} - \lambda W/D)\right] - \frac{hD}{\lambda}T - PIDT\left(M - \frac{T}{2}\right) \tag{33}$$

and

$$g_2(T) = T. \tag{34}$$

Taking the first-order and second-order derivatives of $f_2(T)$, we have

$$f_2'(T) = \frac{D(C\lambda + k)}{\lambda}e^{\lambda T} - \frac{D(k-h)}{\lambda}\frac{e^{\lambda T}}{e^{\lambda T} - \lambda W/D} - \frac{hD}{\lambda} + PID(T - M), \tag{35}$$

and

$$f_2''(T) = D(C\lambda + k)e^{\lambda T} + \frac{D(k-h)(\lambda W/D)e^{\lambda T}}{(e^{\lambda T} - \lambda W/D)^2} + PID > 0. \tag{36}$$

Therefore, $q_2(T) = \frac{f_2(T)}{g_2(T)} = TVC_2(T)$ is a strictly pseudoconvex function in T. Again, from Equation (23), let

$$f_3(T) = S + \frac{CD}{\lambda}(e^{\lambda T} - 1) + \frac{hD}{\lambda^2}[e^{\lambda T} - \lambda T - 1] + \frac{CRD}{\lambda^2}[e^{\lambda(T-M)} - \lambda(T - M) - 1] - PI\frac{DM^2}{2} \tag{37}$$

and

$$g_3(T) = T. \tag{38}$$

Taking the first-order and second-order derivatives of $f_3(T)$, we have

$$f_3'(T) = CDe^{\lambda T} + \frac{hD}{\lambda}[e^{\lambda T} - 1] + \frac{CRD}{\lambda}\left[e^{\lambda(T-M)} - 1\right], \tag{39}$$

and

$$f_3''(T) = CD\lambda e^{\lambda T} + hDe^{\lambda T} + CRDe^{\lambda(T-M)} > 0. \tag{40}$$

Therefore, $q_3(T) = \frac{f_3(T)}{g_3(T)} = TVC_3(T)$ is a strictly pseudoconvex function in T. From Equation (24), let

$$f_4(T) = S + \frac{CD}{\lambda}(e^{\lambda T} - 1) + \frac{hD}{\lambda^2}[e^{\lambda T} - \lambda T - 1] - PIDT\left(M - \frac{T}{2}\right), \tag{41}$$

and

$$g_4(T) = T. \tag{42}$$

Taking the first-order and second-order derivatives of $f_4(T)$, we have

$$f_4'(T) = CDe^{\lambda T} + \frac{hD}{\lambda}[e^{\lambda T} - 1] + PID(T - M), \tag{43}$$

and

$$f_4''(T) = \lambda CDe^{\lambda T} + hDe^{\lambda T} + PID > 0. \tag{44}$$

Therefore, $q_4(T) = \frac{f_4(T)}{g_4(T)} = TVC_4(T)$ is a strictly pseudoconvex function in T. From Equation (25), let

$$f_5(T) = S + \frac{CD}{\lambda}(e^{\lambda T} - 1) + \frac{D(h + CR)}{\lambda^2}[e^{\lambda T} - \lambda T - 1], \tag{45}$$

and

$$g_5(T) = T. \tag{46}$$

Taking the first-order and second-order derivatives of $f_5(T)$, we have

$$f_5'(T) = CDe^{\lambda T} + \frac{D(h + CR)}{\lambda}[e^{\lambda T} - 1], \tag{47}$$

and

$$f_5''(T) = \lambda CDe^{\lambda T} + D(h + CR)e^{\lambda T} > 0. \tag{48}$$

Therefore, $q_5(T) = \frac{f_5(T)}{g_5(T)} = TVC_5(T)$ is a strictly pseudoconvex function in T. Finally, from Equation (27), let

$$\begin{aligned} f_6(T) = S + A + \frac{D(C\lambda + k)}{\lambda^2}(e^{\lambda T} - 1) - \frac{D(k - h)}{\lambda^2}\left[\frac{\lambda}{D}W + \ln(e^{\lambda T} - \lambda W/D)\right] \\ - \frac{hD}{\lambda}T + \frac{CRD}{\lambda^2}[e^{\lambda T} - \lambda T - 1] \end{aligned} \tag{49}$$

and

$$g_6(T) = T. \tag{50}$$

Taking the first-order and second-order derivatives of $f_6(T)$, we have

$$f_6'(T) = \frac{D(C\lambda + k)}{\lambda}e^{\lambda T} - \frac{D(k - h)}{\lambda}\frac{e^{\lambda T}}{e^{\lambda T} - \lambda W/D} - \frac{hD}{\lambda} + \frac{CRD}{\lambda}[e^{\lambda T} - 1], \tag{51}$$

and

$$f_6''(T) = D(C\lambda + k)e^{\lambda T} + \frac{D(k - h)(\lambda W/D)e^{\lambda T}}{(e^{\lambda T} - \lambda W/D)^2} + CRDe^{\lambda T} > 0. \tag{52}$$

Therefore, $q_6(T) = \frac{f_6(T)}{g_6(T)} = TVC_6(T)$ is a strictly pseudoconvex function in T. Consequently, we have completed the proof.

Note that Theorem 1 simplifies the search for the global minimum solution to a local minimum solution. To find the optimal solution T_i^* for $i = 1, 2, \ldots, 6$, taking the first-order derivative of

$TVC_i(T)$, for $i = 1, 2, \ldots, 6$, setting the result to zero, and simplifying terms, we obtain

$$\frac{D(C\lambda + k)}{\lambda^2}(\lambda T e^{\lambda T} - e^{\lambda T} + 1) + \frac{D(k-h)}{\lambda^2}\left[\frac{\lambda}{D}W + \ln(e^{\lambda T} - \lambda W/D) - \frac{\lambda T e^{\lambda T}}{e^{\lambda T} - \lambda W/D}\right]$$
$$+ \frac{CRD}{\lambda^2}[\lambda T e^{\lambda(T-M)} - e^{\lambda(T-M)} - \lambda M + 1] + \frac{PIDM^2}{2} - (S + A) = 0, \quad \text{for } T_1^*; \tag{53}$$

$$\frac{D(C\lambda + k)}{\lambda^2}(\lambda T e^{\lambda T} - e^{\lambda T} + 1) + \frac{D(k-h)}{\lambda^2}\left[\frac{\lambda}{D}W + \ln(e^{\lambda T} - \lambda W/D) - \frac{\lambda T e^{\lambda T}}{e^{\lambda T} - \lambda W/D}\right]$$
$$+ \frac{PIDT^2}{2} - (S + A) = 0, \quad \text{for } T_2^*; \tag{54}$$

$$\frac{D(C\lambda + h)}{\lambda^2}(\lambda T e^{\lambda T} - e^{\lambda T} + 1) + \frac{CRD}{\lambda^2}[\lambda T e^{\lambda(T-M)} - e^{\lambda(T-M)} - \lambda M + 1]$$
$$+ \frac{PIDM^2}{2} - S = 0, \quad \text{for } T_3^*; \tag{55}$$

$$\frac{D(C\lambda + h)}{\lambda^2}(\lambda T e^{\lambda T} - e^{\lambda T} + 1) + \frac{PIDT^2}{2} - S = 0, \quad \text{for } T_4^*; \tag{56}$$

$$\frac{D(C\lambda + h + CR)}{\lambda^2}(\lambda T e^{\lambda T} - e^{\lambda T} + 1) - S = 0, \quad \text{for } T_5^*; \tag{57}$$

and

$$\frac{D(C\lambda + k + CR)}{\lambda^2}(\lambda T e^{\lambda T} - e^{\lambda T} + 1) + \frac{D(k-h)}{\lambda^2}\left[\frac{\lambda}{D}W + \ln(e^{\lambda T} - \lambda W/D) - \frac{\lambda T e^{\lambda T}}{e^{\lambda T} - \lambda W/D}\right]$$
$$- (S + A) = 0, \quad \text{for } T_6^*. \tag{58}$$

In the next section, we provide a couple of examples to compare the optimal solutions in Liao et al. (2012) with ours.

5. Numerical examples

Example 1: We consider the data same as those in Example 5 in Liao et al. (2012): $h = 5$, $k = 6$, $R = 0.15$, $I = 0.12$, $S = 40$, $A = 5$, $C = 3$, $\lambda = 0.03$, $D = 30$, $M = 0.1$, $W = 12$, $\overline{W} = 9$, and we set $P = 10$. In the example, $W > \overline{W}$, the calculation results of Liao et al. (2012) and ours are shown in Table 1.

Table 1
Optimal solution for Example 1

	t_W	t_R	T^*	Q^*	$TVC(T^*)$
Liao et al. (2012)	0.2881	—	0.6857	20.7846	214.4331
Present paper	—	0.2950	0.6891	20.8881	213.3376

Table 2
Optimal solution for Example 2

	t_W	t_R	T^*	Q^*	$TVC(T^*)$
Liao et al. (2012)	0.5013	—	0.7504	15.1779	154.9769
Present paper	—	0.5047	0.7500	15.1699	153.2564

Example 2: We consider the data same as those in Example 4 in Liao et al. (2012): $h = 3$, $k = 5$, $R = 0.15$, $I = 0.12$, $S = 30$, $A = 2$, $C = 4$, $\lambda = 0.03$, $D = 20$, $M = 0.3$, $W = 5$, $\overline{W} = 10$, and we set $P = 10$. In the example, $W < \overline{W}$, the calculation results of Liao et al. (2012) and ours are shown in Table 2.

The computational results in Tables 1 and 2 reveal that (1) the time that inventory level in RW reaches to zero, t_R, is longer than the time that inventory level reduces to W, t_W; and (2) the annual total variable cost, $TVC(T^*)$, in ours is less expensive to operate than that in Liao et al. (2012).

6. Conclusions

In this paper, we have identified the flaws of the paper authored by Liao et al. in 2012 as follows: (1) inclusion of cost of deteriorating items twice, (2) use of unit purchase cost instead of unit selling price to calculate interest earned, and (3) erroneously considering the time taken by inventory in RW to reduce to zero, to be same as the time taken by total inventory to reduce to the maximum storage capacity of the OW. Then we have established appropriately objective function and mathematical derivations of the problem. Further, we have shown that the optimal replenishment cycle time not only exists but is also unique, which simplifies the search for the global minimum solution to a local minimum solution. Finally, we have provided a couple of examples to illustrate the differences between our optimal solution and theirs.

For future research, the proposed model can be extended in several ways. For instance, we may consider this model for time-varying or stock-dependent demand. Also, we could generalize the model to allow for shortages and partial backlogging. Finally, we could consider noncooperative Nash and Stackelberg equilibrium solutions or cooperative Pareto solution to this supplier–retailer supply chain management problem.

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